Gainers and Losers in Priority Services

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Supplementary material.

This appendix contains derivations of Examples 4 and 5, and proofs of Propositions 5, 8, 9, 10, 12 and 15.

Proof of Proposition 5. We start with the case where the provider offers regular service only at price p. Assume that consumers with waiting costs below c^r join the queue while the rest doesn't. Since $c^r < \tilde{v} < \bar{c} = 1$ we have partial coverage of the market (see Lemma 2). The utility of consumer with waiting costs c from joining the service is given by

$$\tilde{v} - p - c\frac{F(c^r)}{2}.$$

Since the utility from not joining is 0 the marginal type c^r should be indifferent between joining the queue and remaining unserved, that is c^r satisfies

$$\tilde{v} - p - c^r\frac{F(c^r)}{2} = 0.$$

The provider's revenues are $pF(c^r)$.

Instead of maximizing over p we can optimize over c^r noting that

$$p = \tilde{v} - c^r\frac{F(c^r)}{2}.$$

Hence the provider's profits are

$$\left(\tilde{v} - c^r\frac{F(c^r)}{2}\right)F(c^r).$$

The FOC is

$$\tilde{v}f(c^r) - \frac{(F(c^r))^2}{2} - c^r F(c^r) f(c^r) = 0.$$

We can look at the FOC wrt c^r because we do not have a full coverage $c^r \in (0,1)$. The optimal cutoff c^r satisfies

$$\tilde{v} = c^r F(c^r) + \frac{(F(c^r))^2}{2f(c^r)}.$$

The consumers' surplus is equal to

$$\begin{aligned}\int_0^{c^r}\left(\tilde{v} - p - c\frac{F(c^r)}{2}\right)f(c)dc &= \int_0^{c^r}\left(c^r\frac{F(c^r)}{2} - c\frac{F(c^r)}{2}\right)f(c)dc \quad (1)\\ &= \frac{F(c^r)}{2}\int_0^{c^r}(c^r - c)f(c)dc\\ &= \frac{F(c^r)}{2}\int_0^{c^r}F(c)dc\end{aligned}$$

where the last equality follows from integration by parts.

Assume now that the provider sets two prices: one for the regular service (π) and one for priority service (Π). Consumers with very low waiting costs choose regular service, consumers with very high waiting costs do not join any service, while consumers in the middle-range join priority service. The utility of consumer with waiting costs c from joining the regular service is

$$\tilde{v}-\pi-c\left[\frac{F\left(c^{I}\right)}{2}+F\left(c^{II}\right)-F\left(c^{I}\right)\right]=\tilde{v}-\pi-c\left[F\left(c^{II}\right)-\frac{F\left(c^{I}\right)}{2}\right]$$

while the utility of consumer with waiting costs c from joining priority service is

$$\tilde{v}-\Pi-c\frac{F\left(c^{II}\right)-F\left(c^{I}\right)}{2}.$$

Type c^{I} is indifferent between getting the regular and the priority services, while type c^{II} is indifferent between getting the priority service and no service at all. That is,

$$\begin{aligned}\tilde{v}-\pi-c^{I}\left[F\left(c^{II}\right)-\frac{F\left(c^{I}\right)}{2}\right] &= \tilde{v}-\Pi-c^{I}\frac{F\left(c^{II}\right)-F\left(c^{I}\right)}{2} \Leftrightarrow \\ \Pi-\pi &= c^{I}\frac{F\left(c^{II}\right)}{2}\end{aligned}$$

and

$$\tilde{v}-\Pi-c^{II}\frac{F\left(c^{II}\right)-F\left(c^{I}\right)}{2}=0.$$

Hence

$$\begin{aligned}\Pi &= \tilde{v}-c^{II}\frac{F\left(c^{II}\right)-F\left(c^{I}\right)}{2} \\ \pi &= \tilde{v}-c^{II}\frac{F\left(c^{II}\right)-F\left(c^{I}\right)}{2}-c^{I}\frac{F\left(c^{II}\right)}{2}.\end{aligned}$$

The seller's expected revenue is

$$\pi F\left(c^{I}\right)+\Pi\left[F\left(c^{II}\right)-F\left(c^{I}\right)\right]$$

while consumers' surplus is

$$\int_{0}^{c^{I}}\left(\tilde{v}-\pi-c\left[F\left(c^{II}\right)-\frac{F\left(c^{I}\right)}{2}\right]\right)f(c)dc+\int_{c^{I}}^{c^{II}}\left(\tilde{v}-\Pi-c\frac{F\left(c^{II}\right)-F\left(c^{I}\right)}{2}\right)f(c)dc.$$

Plugging in the expressions for the prices into the consumers' surplus we get

$$\int_0^{c^I}\left(\tilde{v}-\pi-c\left[F\left(c^{II}\right)-\frac{F\left(c^I\right)}{2}\right]\right)f(c)dc+\int_{c^I}^{c^{II}}\left(\tilde{v}-\Pi-c\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}\right)f(c)dc \quad (2)$$

$$= \int_0^{c^I}\left(c^{II}\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}+c^I\frac{F\left(c^{II}\right)}{2}-c\left[F\left(c^{II}\right)-\frac{F\left(c^I\right)}{2}\right]\right)f(c)dc+$$

$$+\int_{c^I}^{c^{II}}\left(c^{II}\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}-c\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}\right)f(c)dc$$

$$= \left[F\left(c^{II}\right)-\frac{F\left(c^I\right)}{2}\right]\int_0^{c^I}\left(c^I-c\right)f(c)dc+\int_0^{c^{II}}\left(c^{II}-c^I\right)\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}f(c)dc$$

$$+\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}\int_{c_1}^{c^{II}}\left(c^{II}-c\right)f(c)dc$$

$$= \left[F\left(c^{II}\right)-\frac{F\left(c^I\right)}{2}\right]\int_0^{c^I}\left(c^I-c\right)f(c)dc+\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}\int_{c^I}^{c^{II}}\left(c^{II}-c\right)f(c)dc$$

$$+\left(c^{II}-c^I\right)\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}F\left(c^I\right)$$

$$= \left[F\left(c^{II}\right)-\frac{F\left(c^I\right)}{2}\right]\int_0^{c^I}F(c)dc+\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}\left[-\left(c^{II}-c^I\right)F\left(c^I\right)+\int_{c^I}^{c^{II}}F(c)dc\right]$$

$$+\left(c^{II}-c^I\right)\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}F\left(c^I\right)$$

$$= \left[F\left(c^{II}\right)-\frac{F\left(c^I\right)}{2}\right]\int_0^{c^I}F(c)dc+\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}\int_{c^I}^{c^{II}}F(c)dc$$

$$= F\left(c^{II}\right)\int_0^{c^I}F(c)dc-\frac{F\left(c^I\right)}{2}\int_0^{c^{II}}F(c)dc+\frac{F\left(c^{II}\right)}{2}\int_{c^I}^{c^{II}}F(c)dc$$

$$= \frac{F\left(c^{II}\right)}{2}\int_0^{c^{II}}F(c)dc+\frac{F\left(c^{II}\right)}{2}\int_0^{c^I}F(c)dc-\frac{F\left(c^I\right)}{2}\int_0^{c^{II}}F(c)dc.$$

We now find the optimal cutoffs. The seller's expected revenue is

$$R=\pi F\left(c^I\right)+\Pi\left[F\left(c^{II}\right)-F\left(c^I\right)\right]$$

$$= \left(\tilde{v}-c^{II}\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}-c^I\frac{F\left(c^{II}\right)}{2}\right)F\left(c^I\right)+\left(\tilde{v}-c^{II}\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}\right)\left[F\left(c^{II}\right)-F\left(c^I\right)\right]$$

$$= \tilde{v}F\left(c^{II}\right)-c^{II}\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}F\left(c^I\right)-c^I\frac{F\left(c^{II}\right)F\left(c^I\right)}{2}-c^{II}\frac{\left(F\left(c^{II}\right)-F\left(c^I\right)\right)^2}{2}$$

$$= \tilde{v}F\left(c^{II}\right)-c^{II}\frac{F\left(c^{II}\right)-F\left(c^I\right)}{2}F\left(c^{II}\right)-c^I\frac{F\left(c^{II}\right)F\left(c^I\right)}{2}.$$

Maximizing it w.r.t. c^I and c^{II} gives the FOC

$$\frac{\partial R}{\partial c^{II}} = \tilde{v} f\left(c^{II}\right) - \frac{F\left(c^{II}\right) - F\left(c^{I}\right)}{2} F\left(c^{II}\right) - c^{II} f\left(c^{II}\right) \frac{F\left(c^{II}\right) - F\left(c^{I}\right)}{2}$$
$$-c^{II} \frac{f\left(c^{II}\right)}{2} F\left(c^{II}\right) - c^{I} \frac{f\left(c^{II}\right) F\left(c^{I}\right)}{2} = 0$$
$$\frac{\partial R}{\partial c^{I}} = c^{II} \frac{f\left(c^{I}\right)}{2} F\left(c^{II}\right) - \frac{F\left(c^{II}\right) F\left(c^{I}\right)}{2} - c^{I} \frac{F\left(c^{II}\right) f\left(c^{I}\right)}{2}$$
$$= \left(c^{II} - c^{I}\right) \frac{f\left(c^{I}\right)}{2} F\left(c^{II}\right) - \frac{F\left(c^{II}\right) F\left(c^{I}\right)}{2} = 0$$

Since $c^{II} = 0$ is not a maximum we get

$$\frac{\partial R}{\partial c^{I}} = 0 \Longleftrightarrow \left(c^{II} - c^{I}\right) f\left(c^{I}\right) - F\left(c^{I}\right) = 0 \Longleftrightarrow c^{II} = c^{I} + \frac{F\left(c^{I}\right)}{f\left(c^{I}\right)}$$

and $\frac{\partial R}{\partial c^{II}} = 0 \Longleftrightarrow$

$$\tilde{v} - \frac{F\left(c^{II}\right) - F\left(c^{I}\right)}{2} \frac{F\left(c^{II}\right)}{f\left(c^{II}\right)} - c^{II} \left[F\left(c^{II}\right) - \frac{F\left(c^{I}\right)}{2}\right] - c^{I} \frac{F\left(c^{I}\right)}{2} = 0$$

Alternative writing $\frac{\partial R}{\partial c^{II}} = 0 \Longleftrightarrow$

$$\tilde{v} - \frac{F\left(c^{II}\right) - F\left(c^{I}\right)}{2} \left[\frac{F\left(c^{II}\right)}{f\left(c^{II}\right)} + c^{II}\right] - c^{II} \frac{F\left(c^{II}\right)}{2} - c^{I} \frac{F\left(c^{I}\right)}{2} = 0.$$

Assume that $F(c) = c^{\mu}$ for $c \in [0, 1]$ and $f(c) = \mu c^{\mu-1}$. Then **without priority service** we have

$$\begin{aligned}
\tilde{v} &= c^{r} F\left(c^{r}\right) + \frac{\left(F\left(c^{r}\right)\right)^{2}}{2 f\left(c^{r}\right)} \Longleftrightarrow \\
\tilde{v} &= c^{r} \left(c^{r}\right)^{\mu} + \frac{\left(c^{r}\right)^{2\mu}}{2\theta \left(c^{r}\right)^{\mu-1}} \Longleftrightarrow \tilde{v} = \left(c^{r}\right)^{\mu+1} + \frac{\left(c^{r}\right)^{\mu+1}}{2\mu} \\
\tilde{v} &= \left(c^{r}\right)^{\mu+1} \left(1 + \frac{1}{2\mu}\right) \Longleftrightarrow \left(c^{r}\right)^{\mu+1} = \frac{2\mu\tilde{v}}{2\mu+1} \Longleftrightarrow c^{r} = \left(\frac{2\mu\tilde{v}}{2\mu+1}\right)^{\frac{1}{\mu+1}}
\end{aligned}$$

Plugging the expression for c^r into the expression for consumers' surplus given in (1) gives

$$\frac{F\left(c^{r}\right)}{2} \int_{0}^{c^{r}} F(c) dc = \frac{\left(c^{r}\right)^{\mu}}{2} \int_{0}^{c^{r}} c^{\mu} dc = \frac{1}{2} \frac{1}{\mu+1} \left(c^{r}\right)^{2\mu+1} = \frac{1}{2} \frac{1}{\mu+1} \left(\frac{2\mu\tilde{v}}{2\mu+1}\right)^{\frac{2\mu+1}{\mu+1}}. \quad (3)$$

While the price is

$$p = \tilde{v} - c^{r} \frac{F\left(c^{r}\right)}{2} = \tilde{v} - \frac{1}{2} \left(\frac{2\mu\tilde{v}}{2\mu+1}\right)^{\frac{1}{\mu+1}} \left(\frac{2\mu\tilde{v}}{2\mu+1}\right)^{\frac{\mu}{\mu+1}} = \tilde{v} - \frac{1}{2} \left(\frac{2\mu\tilde{v}}{2\mu+1}\right).$$

With priority service we get

$$c^{II} = c^I + \frac{c^I}{\mu} \Longleftrightarrow c^{II} = c^I \frac{\mu+1}{\mu}$$

and

$$\begin{aligned}
\tilde{v} - \frac{\left(c^{II}\right)^{\mu} - \left(c^{I}\right)^{\mu}}{2}\left[c^{II} + \frac{c^{II}}{\mu}\right] - \frac{\left(c^{II}\right)^{\mu+1}}{2} - \frac{\left(c^{I}\right)^{\mu+1}}{2} &= 0 \Longleftrightarrow \\
\tilde{v} - \frac{\left(c^{II}\right)^{\mu} - \left(c^{I}\right)^{\mu}}{2} c^{II} \frac{\mu+1}{\mu} - \frac{\left(c^{II}\right)^{\mu+1}}{2} - \frac{\left(c^{I}\right)^{\mu+1}}{2} &= 0 \Longleftrightarrow \\
\tilde{v} - \frac{\left(c^{II}\right)^{\mu+1}}{2} \frac{2\mu+1}{\mu} + \frac{\left(c^{I}\right)^{\mu}}{2} c^{II} \frac{\mu+1}{\mu} - \frac{\left(c^{I}\right)^{\mu+1}}{2} &= 0
\end{aligned}$$

Plugging the expression we got for c^{II} the last equation becomes

$$\begin{aligned}
\tilde{v} - \frac{\left(c^{I}\right)^{\mu+1}}{2} \left(\frac{\mu+1}{\mu}\right)^{\mu+1} \frac{2\mu+1}{\mu} + \frac{\left(c^{I}\right)^{\mu+1}}{2} \left(\frac{\mu+1}{\mu}\right)^{2} - \frac{\left(c^{I}\right)^{\mu+1}}{2} = 0 \Longleftrightarrow \\
\tilde{v} = \frac{\left(c^{I}\right)^{\mu+1}}{2} \left[\left(\frac{\mu+1}{\mu}\right)^{\mu+1} \frac{2\mu+1}{\mu} - \left(\frac{\mu+1}{\mu}\right)^{2} + 1\right] \Longleftrightarrow \\
2\tilde{v} = \left(c^{I}\right)^{\mu+1} \left[\left(\frac{\mu+1}{\mu}\right)^{\mu+1} \frac{2\mu+1}{\mu} - \frac{2\mu+1}{\mu^2}\right] \Longleftrightarrow \\
2\tilde{v} = \left(c^{I}\right)^{\mu+1} \left[\left(\frac{\mu+1}{\mu}\right)^{\mu+1} - \frac{1}{\mu}\right] \frac{2\mu+1}{\mu} \Longleftrightarrow \\
\frac{2\mu\tilde{v}}{(2\mu+1)\left[\left(\frac{\mu+1}{\mu}\right)^{\mu+1} - \frac{1}{\mu}\right]} = \left(c^{I}\right)^{\mu+1} \Longleftrightarrow \\
c^{I} = \left(\frac{2\mu\tilde{v}}{(2\mu+1)\left[\left(\frac{\mu+1}{\mu}\right)^{\mu+1} - \frac{1}{\mu}\right]}\right)^{\frac{1}{\mu+1}} \\
c^{II} = \frac{\mu+1}{\mu}\left(\frac{2\mu\tilde{v}}{(2\mu+1)\left[\left(\frac{\mu+1}{\mu}\right)^{\mu+1} - \frac{1}{\mu}\right]}\right)^{\frac{1}{\mu+1}}
\end{aligned}$$

Plugging the expressions for c^I and c^{II} into the expression for consumers' surplus given in (2) gives

$$\begin{aligned}
&\frac{F\left(c^{II}\right)}{2}\int_0^{c^{II}}F(c)dc+\frac{F\left(c^{II}\right)}{2}\int_0^{c^{I}}F(c)dc-\frac{F\left(c^{I}\right)}{2}\int_0^{c^{II}}F(c)dc \qquad (4)\\
=\;&\frac{1}{2}\left(\left(c^{II}\right)^{\mu}\int_0^{c^{II}}(c)^{\mu}dc+\left(c^{II}\right)^{\mu}\int_0^{c^{I}}(c)^{\mu}dc-\left(c^{I}\right)^{\mu}\int_0^{c^{II}}(c)^{\mu}dc\right)\\
=\;&\frac{1}{2}\frac{1}{\mu+1}\left(\left(c^{II}\right)^{2\mu+1}+\left(c^{II}\right)^{\mu}\left(c^{I}\right)^{\mu+1}-\left(c^{I}\right)^{\mu}\left(c^{II}\right)^{\mu+1}\right)\\
=\;&\frac{1}{2}\frac{1}{\mu+1}\left(c^{I}\right)^{2\mu+1}\left(\left(\frac{\mu+1}{\mu}\right)^{2\mu+1}+\left(\frac{\mu+1}{\mu}\right)^{\mu}-\left(\frac{\mu+1}{\mu}\right)^{\mu+1}\right)\\
=\;&\frac{1}{2}\frac{1}{\mu+1}\left(\frac{2\mu\tilde{v}}{(2\mu+1)\left[\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right]}\right)^{\frac{2\mu+1}{\mu+1}}\left(\frac{\mu+1}{\mu}\right)^{\mu}\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right)\\
=\;&\left(\frac{2\mu\tilde{v}}{(2\mu+1)}\right)^{\frac{2\mu+1}{\mu+1}}\frac{\left(\frac{\mu+1}{\mu}\right)^{\mu-1}\frac{1}{2\mu}}{\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right)^{\frac{\mu}{\mu+1}}}.
\end{aligned}$$

We show now that the consumers' surplus if no priority service is offered (3) exceeds the one if it is offered (4):We will show now that

$$\left(\frac{2\mu\tilde{v}}{(2\mu+1)}\right)^{\frac{2\mu+1}{\mu+1}}\frac{\left(\frac{\mu+1}{\mu}\right)^{\mu-1}\frac{1}{2\mu}}{\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right)^{\frac{\mu}{\mu+1}}}>\frac{1}{2}\frac{1}{\mu+1}\left(\frac{2\mu\tilde{v}}{2\mu+1}\right)^{\frac{2\mu+1}{\mu+1}}\iff$$

$$\frac{\left(\frac{\mu+1}{\mu}\right)^{\mu-1}\frac{1}{2\mu}}{\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right)^{\frac{\mu}{\mu+1}}}>\frac{1}{2}\frac{1}{\mu+1}\iff$$

$$\frac{\left(\frac{\mu+1}{\mu}\right)^{\mu-1}\frac{\mu+1}{\mu}}{\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right)^{\frac{\mu}{\mu+1}}}>1\iff$$

$$\left(\frac{\mu+1}{\mu}\right)^{\mu}>\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right)^{\frac{\mu}{\mu+1}}\iff$$

$$\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}\right)^{\frac{\mu}{\mu+1}}>\left(\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}\right)^{\frac{\mu}{\mu+1}}\iff$$

$$\left(\frac{\mu+1}{\mu}\right)^{\mu+1}>\left(\frac{\mu+1}{\mu}\right)^{\mu+1}-\frac{1}{\mu}$$

which holds for any $\mu > 0$. ■

Proof of Proposition 8. Our proof is recursive. We will show that for any number of priority classes l and cutoffs $\{c_1, ..., c_{l-1}\}$ when $c_{l-1} < \bar{c}$, adding another (higher) priority class and cutoff $c_l \in (c_{l-1}, \bar{c})$ decreases the consumers' welfare. Since we showed that adding a single priority class decreases the consumers' welfare, using this argument repeatedly (while each time adding another, higher priority class) allows us to conclude that any selling procedure that involves $k > 1$ priority classes with cutoffs $\{c_1, ..., c_{k-1}\}$ is inferior to no priority from the consumer welfare perspective.[1]

We will show now that adding a higher priority class decreases the consumers' welfare. If there are l priority classes with cutoffs $\{c_1, ..., c_{l-1}\}$ and $c_{l-1} < \bar{c}$, then the prices of p_l and p_{l-1} are such that the type c_{l-1} is indifferent between joining either of the two highest classes – either priority class l or $l-1$. That is,

$$\begin{aligned} -p_l - \frac{1-F(c_{l-1})}{2} c_{l-1} &= -p_{l-1} - \left(1 - F(c_{l-1}) + \frac{F(c_{l-1}) - F(c_{l-2})}{2}\right) c_{l-1} \Longleftrightarrow \\ p_l &= p_{l-1} + \frac{1-F(c_{l-2})}{2} c_{l-1}. \end{aligned}$$

The consumers' welfare is given by

$$\begin{aligned} CS(&\leq c_{l-1}) + \int_{c_{l-1}}^{\bar{c}} \left[-p_l - \frac{1-F(c_{l-1})}{2} c\right] f(c)dc \\ &= CS(\leq c_{l-1}) - p_{l-1}\left[1 - F(c_{l-1})\right] - \frac{1-F(c_{l-2})}{2} c_{l-1}\left[1-F(c_{l-1})\right] \\ &\quad - \frac{1-F(c_{l-1})}{2} \int_{c_{l-1}}^{\bar{c}} cf(c)dc \end{aligned}$$

where $CS(\leq c_{l-1})$ is the consumers' welfare of all types lower than c_{l-1}.

If there are $l+1$ priority classes with cutoffs $\{c_1, ..., c_{l-1}, c_l\}$ and $c_l \in (c_{l-1}, \bar{c})$ (so that the additional class partitions the interval of the previously highest class into two intervals), then the price p'_l is such that type c_{l-1} is indifferent between priority classes l and $l-1$, while the price p'_{l+1} is such that type c_l is indifferent between priority classes l and $l+1$. Observe that this change does not affect the prices of the lower classes. The first indifference condition is

$$\begin{aligned} -p_{l-1} - \left(1 - F(c_{l-1}) + \frac{F(c_{l-1}) - F(c_{l-2})}{2}\right) c_{l-1} &= -p'_l - \left(1 - F(c_l) + \frac{F(c_l) - F(c_{l-1})}{2}\right) c_{l-1} \Longleftrightarrow \\ p'_l &= p_{l-1} + \frac{F(c_l) - F(c_{l-2})}{2} c_{l-1}. \end{aligned}$$

[1] If $c_{l-1} = \bar{c}$, then applying this construction to the highest cutoff below $\bar{c}$ allows us to make the same conclusion.

Type c_k must be indifferent between classes k and $k+1$:

$$-p_l' - \left(1 - F\left(c_l\right) + \frac{F\left(c_l\right) - F\left(c_{l-1}\right)}{2}\right) c_l = -p_{l+1}' - \frac{1 - F\left(c_l\right)}{2} c_l \Longleftrightarrow$$

$$p_{l+1}' = p_l' + \frac{1 - F\left(c_{l-1}\right)}{2} c_l = p_{l-1} + \frac{F\left(c_l\right) - F\left(c_{l-2}\right)}{2} c_{l-1} + \frac{1 - F\left(c_{l-1}\right)}{2} c_l$$

The consumers' welfare is

$$CS(\leq c_{l-1}) + \int_{c_{l-1}}^{c_l} \left[-p_l' - \left(1 - F\left(c_l\right) + \frac{F\left(c_l\right) - F\left(c_{l-1}\right)}{2}\right) c\right] f(c)dc$$
$$+ \int_{c_l}^{\overline{c}} \left[-p_{l+1}' - \frac{1 - F\left(c_l\right)}{2} c\right] f(c)dc$$
$$= CS(\leq c_{l-1}) + \int_{c_{l-1}}^{c_l} \left[-p_{l-1} - \frac{F\left(c_l\right) - F\left(c_{l-2}\right)}{2} c_{l-1} - \left(1 - F\left(c_l\right) + \frac{F\left(c_l\right) - F\left(c_{l-1}\right)}{2}\right) c\right] f(c)dc$$
$$+ \int_{c_l}^{\overline{c}} \left[-p_{l-1} - \frac{F\left(c_l\right) - F\left(c_{l-2}\right)}{2} c_{l-1} - \frac{1 - F\left(c_{l-1}\right)}{2} c_l - \frac{1 - F\left(c_l\right)}{2} c\right] f(c)dc$$
$$= CS(\leq c_{l-1}) - p_{l-1}\left[1 - F\left(c_{l-1}\right)\right] - \frac{F\left(c_l\right) - F\left(c_{l-2}\right)}{2} c_{l-1}\left[F\left(c_l\right) - F\left(c_{l-1}\right)\right]$$
$$- \left(1 - \frac{F\left(c_l\right)}{2} - \frac{F\left(c_{l-1}\right)}{2}\right) \int_{c_{l-1}}^{c_l} cf(c)dc - \frac{F\left(c_l\right) - F\left(c_{l-2}\right)}{2} c_{l-1}\left[1 - F\left(c_l\right)\right]$$
$$- \frac{1 - F\left(c_{l-1}\right)}{2} c_l \left[1 - F\left(c_l\right)\right] - \frac{1 - F\left(c_l\right)}{2} \int_{c_l}^{\overline{c}} cf(c)dc$$
$$= CS(\leq c_{l-1}) - p_{l-1}\left[1 - F\left(c_{l-1}\right)\right] - \frac{F\left(c_l\right) - F\left(c_{l-2}\right)}{2} c_{l-1}\left[1 - F\left(c_{l-1}\right)\right]$$
$$- \left(\frac{1 - F\left(c_l\right)}{2} + \frac{1 - F\left(c_{l-1}\right)}{2}\right) \int_{c_{l-1}}^{c_l} cf(c)dc - \frac{1 - F\left(c_{l-1}\right)}{2} c_l \left[1 - F\left(c_l\right)\right] - \frac{1 - F\left(c_l\right)}{2} \int_{c_l}^{\overline{c}} cf(c)dc$$
$$= CS(\leq c_{l-1}) - p_{l-1}\left[1 - F\left(c_{l-1}\right)\right] - \frac{F\left(c_l\right) - F\left(c_{l-2}\right)}{2} c_{l-1}\left[1 - F\left(c_{l-1}\right)\right] - \frac{1 - F\left(c_{l-1}\right)}{2} \int_{c_{l-1}}^{\overline{c}} cf(c)dc$$
$$- \frac{1 - F\left(c_l\right)}{2} \int_{c_{l-1}}^{c_l} cf(c)dc - \frac{1 - F\left(c_{l-1}\right)}{2} c_l \left[1 - F\left(c_l\right)\right] - \frac{F\left(c_{l-1}\right) - F\left(c_l\right)}{2} \int_{c_l}^{\overline{c}} cf(c)dc$$

Introducing an additional priority class decreases the consumers' surplus if and only if

$$CS(\leq c_{l-1}) - p_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{1-F\left(c_{l-2}\right)}{2}c_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{1-F\left(c_{l-1}\right)}{2}\int_{c_{l-1}}^{\overline{c}} cf(c)dc >$$

$$CS(\leq c_{l-1}) - p_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{F\left(c_{l}\right)-F\left(c_{l-2}\right)}{2}c_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{1-F\left(c_{l-1}\right)}{2}\int_{c_{l-1}}^{\overline{c}} cf(c)dc$$

$$-\frac{1-F\left(c_{l}\right)}{2}\int_{c_{l-1}}^{c_l} cf(c)dc - \frac{1-F\left(c_{l-1}\right)}{2}c_l\left[1-F\left(c_{l}\right)\right] + \frac{F\left(c_{l}\right)-F\left(c_{l-1}\right)}{2}\int_{c_l}^{\overline{c}} cf(c)dc \Leftrightarrow$$

$$-\frac{1-F\left(c_{l-2}\right)}{2}c_{l-1}\left[1-F\left(c_{l-1}\right)\right] > -\frac{F\left(c_{l}\right)-F\left(c_{l-2}\right)}{2}c_{l-1}\left[1-F\left(c_{l-1}\right)\right]$$

$$-\frac{1-F\left(c_{l}\right)}{2}\int_{c_{l-1}}^{c_l} cf(c)dc - \frac{1-F\left(c_{l-1}\right)}{2}c_l\left[1-F\left(c_{l}\right)\right] + \frac{F\left(c_{l}\right)-F\left(c_{l-1}\right)}{2}\int_{c_l}^{\overline{c}} cf(c)dc.$$

Observe that for either $c_l = c_{l-1}$ or $c_l = \overline{c}$ we have equality in the previous expression. The derivative of the right-hand side of the last inequality with respect to c_l equals

$$\begin{aligned}
& -f\left(c_l\right)c_{l-1}\frac{1-F\left(c_{l-1}\right)}{2} + \frac{f\left(c_l\right)}{2}\int_{c_{l-1}}^{c_l} cf(c)dc - \frac{1-F\left(c_l\right)}{2}c_l f\left(c_l\right) - \frac{1-F\left(c_{l-1}\right)}{2}\left[1-F\left(c_l\right)\right] \\
& +\frac{1-F\left(c_{l-1}\right)}{2}c_l f\left(c_l\right) + \frac{f\left(c_l\right)}{2}\int_{c_l}^{\overline{c}} cf(c)dc - \frac{F\left(c_l\right)-F\left(c_{l-1}\right)}{2}c_l f\left(c_l\right) \\
= \; & -f\left(c_l\right)c_{l-1}\frac{1-F\left(c_{l-1}\right)}{2} + \frac{f\left(c_l\right)}{2}\int_{c_{l-1}}^{\overline{c}} cf(c)dc - \frac{1-F\left(c_{l-1}\right)}{2}\left[1-F\left(c_l\right)\right] \\
= \; & f\left(c_l\right)\left[-c_{l-1}\frac{1-F\left(c_{l-1}\right)}{2} + \frac{1}{2}\int_{c_{l-1}}^{\overline{c}} cf(c)dc - \frac{1-F\left(c_{l-1}\right)}{2}\frac{1-F\left(c_l\right)}{f\left(c_l\right)}\right],
\end{aligned}$$

while the derivative of the left-hand side is zero. Observe that IFR implies that the derivative that we calculated in the last expression changes its sign once from negative to positive, and hence for any $c_l \in (c_{l-1}, \overline{c})$ it holds that

$$CS(\leq c_{l-1}) - p_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{1-F\left(c_{l-2}\right)}{2}c_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{1-F\left(c_{l-1}\right)}{2}\int_{c_{l-1}}^{\overline{c}} cf(c)dc >$$

$$CS(\leq c_{l-1}) - p_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{F\left(c_{l}\right)-F\left(c_{l-2}\right)}{2}c_{l-1}\left[1-F\left(c_{l-1}\right)\right] - \frac{1-F\left(c_{l-1}\right)}{2}\int_{c_{l-1}}^{\overline{c}} cf(c)dc.$$

■

Proof of Proposition 9. Assume that $\{c_1, c_2, ..., c_{k-1}\}$ are the profit-maximizing cutoffs in the problem with k priority classes where $c_{k-1} < \overline{c}$. In the problem with $k+1$ classes set cutoffs $\{c_1', c_2', ..., c_{k-1}', c_k\}$ such that $c_i' = c_i$ for any $i \in \{1, ..., k-1\}$ and $c_k \in (c_{k-1}, \overline{c})$ (in case $c_{k-1} = \overline{c}$ choose the highest cutoff below $\overline{c}$ and split the interval between this cutoff and $\overline{c}$ into two intervals). The allocation induced by cutoffs $\{c_1', c_2', ..., c_{k-1}', c_k\}$ generates higher total welfare than the allocation $\{c_1, c_2, ..., c_{k-1}\}$ as the allocation $\{c_1', c_2', ..., c_{k-1}', c_k\}$ has lower overall waiting costs. The proof of Proposition 8 implies that the consumers' welfare in the allocation induced by $\{c'_1, c'_2, ..., c'_{k-1}, c_k\}$ is lower than in the one induced by $\{c_1, c_2, ..., c_{k-1}\}$. Hence, the provider's profits are higher. Optimizing over the cutoffs $\{c'_1, c'_2, ..., c'_{k-1}, c_k\}$ further increases the provider's profits. Moreover, this reoptimization over the cutoffs does not eliminate any priority classes as otherwise it would allow a further increase in the provider's profits by repeating the first argument of this proof (i.e., adding the eliminated class by splitting the interval of the highest priority class). ■

Derivations for Example 4. The first-order condition of maximizing the monopolist's revenue with respect to c_i for $i \in \{c_1, ..., c_k\}$ is

$$\frac{(1 - F(c_i))(F(c_{i+1}) - F(c_{i-1}))}{2} - f(c_i)\frac{F(c_{i+1}) - F(c_{i-1})}{2}c_i$$
$$+f(c_i)\frac{1 - F(c_{i-1})}{2}c_{i-1} - f(c_i)\frac{1 - F(c_{i+1})}{2}c_{i+1} = 0$$

We can rewrite it as follows

$$(1 - F(c_i)) = f(c_i)c_i - f(c_i)\frac{1 - F(c_{i-1})}{F(c_{i+1}) - F(c_{i-1})}c_{i-1} + f(c_i)\frac{1 - F(c_{i+1})}{F(c_{i+1}) - F(c_{i-1})}c_{i+1}$$

For the uniform distribution with $[0, 1]$ support we get

$$1 - c_i = c_i - \frac{1 - c_{i-1}}{c_{i+1} - c_{i-1}}c_{i-1} + \frac{1 - c_{i+1}}{c_{i+1} - c_{i-1}}c_{i+1}.$$

The solution to the last equations is

$$c_i = \frac{i}{k}$$

The optimal prices are

$$p_i = \sum_{j=1}^{i-1}\frac{F(c_{j+1}) - F(c_{j-1})}{2}c_j = \sum_{j=1}^{i-1}\frac{c_{j+1} - c_{j-1}}{2}c_j = \sum_{j=1}^{i-1}\frac{i}{k^2} = \frac{i(i-1)}{2k^2}$$

and the optimal revenues are

$$\begin{aligned} R &= \sum_{i=2}^{k} p_i [F(c_i) - F(c_{i-1})] = \sum_{i=1}^{k}\frac{i(i-1)}{2k^2}\frac{1}{k} = \frac{1}{2k^3}\sum_{i=1}^{k}\left(i^2 - i\right) \\ &= \frac{1}{2k^3}\left(\frac{k(k+1)(2k+1)}{6} - \frac{k(1+k)}{2}\right) = \frac{1}{6}\left(1 - \frac{1}{k^2}\right) \end{aligned}$$

The consumer's welfare consists of two parts: (1) the increase in the welfare due to a more efficient allocation and (2) the decrease in the welfare due to monetary transfer to the provider. The second part is equal to the revenue of the provider.

The aggregated welfare from the allocation for the cutoffs $c_1, ..., c_{k-1}$ is given by

$$\begin{aligned}
& -\int_{c_{k-1}}^{\bar{c}} \frac{1-F(c_{k-1})}{2} cf(c)\,dc - \int_{c_{k-2}}^{c_{k-1}} \left(1 - F(c_{k-1}) + \frac{F(c_{k-1}) - F(c_{k-2})}{2}\right) cf(c)\,dc \\
& -\int_{c_{k-3}}^{c_{k-2}} \left(1 - F(c_{k-2}) + \frac{F(c_{k-2}) - F(c_{k-3})}{2}\right) cf(c)\,dc \\
& -... - \int_{0}^{c_1} \left(1 - F(c_1) + \frac{F(c_1) - F(c_0)}{2}\right) cf(c)\,dc \\
= & -\int_{c_{k-1}}^{\bar{c}} \left(1 - \frac{F(\bar{c}) + F(c_{k-1})}{2}\right) cf(c)\,dc - \int_{c_{k-2}}^{c_{k-1}} \left(1 - \frac{F(c_{k-1}) + F(c_{k-2})}{2}\right) cf(c)\,dc \\
& -\int_{c_{k-3}}^{c_{k-2}} \left(1 - \frac{F(c_{k-2}) + F(c_{k-3})}{2}\right) cf(c)\,dc - ... - \int_{0}^{c_1} \left(1 - \frac{F(c_1) + F(c_0)}{2}\right) cf(c)\,dc \\
= & -\int_{0}^{\bar{c}} cf(c)\,dc + \int_{c_{k-1}}^{\bar{c}} \frac{F(\bar{c}) + F(c_{k-1})}{2} cf(c)\,dc + \int_{c_{k-2}}^{c_{k-1}} \frac{F(c_{k-1}) + F(c_{k-2})}{2} cf(c)\,dc \\
& + \int_{c_{k-3}}^{c_{k-2}} \frac{F(c_{k-2}) + F(c_{k-3})}{2} cf(c)\,dc + ... + \int_{0}^{c_1} \frac{F(c_1) + F(c_0)}{2} cf(c)\,dc \\
= & -\mathbb{E}(c) + \frac{1}{2} \int_{c_{k-1}}^{\bar{c}} cf(c)\,dc + \sum_{i=1}^{k-1} \frac{F(c_{k-i})}{2} \int_{c_{k-i-1}}^{c_{k-i+1}} cf(c)\,dc.
\end{aligned}$$

Plugging the expressions of the optimal cutoffs for the uniform distribution gives us the customers' welfare from the improved allocation is

$$-\frac{1}{2} + \frac{1}{3} - \frac{1}{12k^2}.$$

Proof of Proposition 10. To characterize the equilibrium of the subgame following the price announcement (p_1, p_2) we will consider a few possible profiles.

Profile (A) $n_1^p > 0$, $n_2^p > 0$, $n_1^{np} > 0$, $n_2^{np} > 0$.
Profile (B) $n_1^p > 0$, $n_2^p > 0$, $n_1^{np} = 0$, $n_2^{np} > 0$.
Profile (C) $n_1^p > 0$, $n_2^p > 0$, $n_1^{np} > 0$, $n_2^{np} = 0$, which is symmetric to profile B.
Profile (D) $n_1^p > 0$, $n_2^p > 0$, $n_1^{np} = 0$, $n_2^{np} = 0$.
Profile (E) $n_1^p > 0$, $n_2^p = 0$, $n_1^{np} > 0$, $n_2^{np} > 0$.
Profile (F) $n_1^p = 0$, $n_2^p > 0$, $n_1^{np} > 0$, $n_2^{np} > 0$, which is symmetric to profile F.
Profile (G) $n_1^p = 0$, $n_2^p = 0$, $n_1^{np} > 0$, $n_2^{np} > 0$.
Profile (H) $n_1^p > 0$, $n_2^p = 0$, $n_1^{np} = 0$, $n_2^{np} > 0$.
Profile (I) $n_1^p = 0$, $n_2^p > 0$, $n_1^{np} > 0$, $n_2^{np} = 0$, which is symmetric to profile H.

There is no equilibrium in which $n_1^p > 0$, $n_2^p = 0$, $n_1^{np} > 0$, $n_2^{np} = 0$ as customers from the regular service of provider 1 should switch to the regular service of provider 2. For a similar reason there is no equilibrium in which $n_1^p = 0$, $n_2^p > 0$, $n_1^{np} = 0$, $n_2^{np} > 0$. Also there is no equilibrium in which all the customers are concentrated at a single provider in either of the services.

We now consider each of the above profiles separately.

Profile A $n_1^p > 0$, $n_2^p > 0$, $n_1^{np} > 0$ and $n_2^{np} > 0$. Since for both providers both classes are nonempty, in this profile the customers are indifferent between all their opportunities (provider 1 vs. provider 2, priority vs. regular services) and the equilibrium conditions are

1. $p_1 = \frac{n_1^p + n_1^{np}}{2}$
2. $p_2 = \frac{n_2^p + n_2^{np}}{2}$
3. $-p_1 - \frac{n_1^p}{2} = -p_2 - \frac{n_2^p}{2}$
4. $n_1^p + n_2^p + n_1^{np} + n_2^{np} = 1$

Observe that (1), (2), and (4) imply that $p_1 + p_2 = \frac{1}{2}$. Further we get $n_2^p + n_2^{np} = 2p_2$ and $n_1^p + n_1^{np} = 2p_1$ and $n_1^p - n_2^p = 2(p_2 - p_1)$, which implies that $n_1^p = 2 - n_2^{np} - 6p_1, n_1^{np} = 8p_1 + n_2^{np} - 2, n_2^p = 1 - n_2^{np} - 2p_1$.

Further observe that $n_1^p = 2 - n_2^{np} - 6p_1 > 0$ implies that $p_1 < \frac{1}{3}$. Symmetry implies that $p_2 < \frac{1}{3}$. In this case we have a continuum of equilibria. Conditions (1)–(3) imply that

$$\begin{aligned} n_1^p &= 2 - n_2^{np} - 6p_1 \\ n_1^{np} &= 8p_1 + n_2^{np} - 2 \\ n_2^p &= 1 - n_2^{np} - 2p_1, \end{aligned}$$

and so for any $\max\{2 - 8p_1, 0\} < n_2^{np} < \min\{2 - 6p_1, 1 - 2p_1\}$ we have an equilibrium. Observe that provider 1 is interested in the lowest possible n_2^{np}. For future derivations, observe that for provider 1, in the best equilibrium n_1^p is (strictly) lower than $2p_1 = 1 - 2p_2$ if $p_1 < 1/4$ and n_1^p is (strictly) lower than $2 - 6p_1 = 6p_2 - 1$ if $p_1 > 1/4$, and the revenues are smaller than $\left(\frac{1}{2} - p_2\right)(1 - 2p_2)$ if $p_1 < 1/4$ (or $p_2 > 1/4$) and smaller than $(6p_2 - 1)\left(\frac{1}{2} - p_2\right)$ if $p_1 > 1/4$ (or $p_2 < 1/4$).

Profile B. Consider the profile with $n_1^p > 0$, $n_2^p > 0$, $n_2^{np} > 0$ and $n_1^{np} = 0$. This profile implies that $n_2 = 2p_2$. For this profile to be part of an equilibrium customers must be indifferent between getting priority service from provider 1 or 2 and regular service from provider 2. These indifference conditions imply

$$\begin{aligned} -p_1 - \frac{n_1^p}{2} &= -p_2 - \frac{n_2^p}{2} \\ p_2 &= \frac{n_2^p + n_2^{np}}{2} \\ 1 &= n_1^p + n_2^p + n_2^{np}. \end{aligned}$$

We have

$$p_2 = \frac{1-n_1^p}{2} \Longleftrightarrow -p_1 - \frac{n_1^p}{2} = -\frac{1-n_1^p}{2} - \frac{n_2^p}{2} \Longleftrightarrow n_1^p = \frac{1}{2} - p_1 + \frac{n_2^p}{2}.$$

Therefore,

$$p_2 = \frac{n_2^p + n_2^{np}}{2} \Longleftrightarrow n_2^p + n_2^{np} = 2p_2 \Longleftrightarrow n_2^{np} = 2p_2 - n_2^p$$

$$\frac{1}{2} - p_1 + \frac{n_2^p}{2} + n_2^p + 2p_2 - n_2^p = 1 \Longleftrightarrow \frac{n_2^p}{2} = \frac{1}{2} + p_1 - 2p_2 \Longleftrightarrow n_2^p = 1 + 2p_1 - 4p_2.$$

This implies that

$$\begin{aligned} n_2^{np} &= 2p_2 - n_2^p = 6p_2 - 1 - 2p_1 \\ n_1^p &= \frac{1}{2} - p_1 + \frac{n_2^p}{2} = 1 - 2p_2 > 0. \end{aligned}$$

In addition, for this profile to be an equilibrium, the utility from joining the regular service of provider 1 must be lower than all other options:

$$p_1 \leq \frac{n_1^p}{2} \Longleftrightarrow 1 - 2p_2 \geq 2p_1.$$

To summarize, this profile is an equilibrium if

$$\begin{aligned} 1 + 2p_1 - 4p_2 &\geq 0 \Longleftrightarrow p_2 \leq \frac{1}{4} + \frac{1}{2}p_1 \\ 1 - 2p_2 - 2p_1 &\geq 0 \Longleftrightarrow p_2 \leq \frac{1}{2} - p_1 \\ 6p_2 - 1 - 2p_1 &\geq 0 \Longleftrightarrow p_2 \geq \frac{1}{6} + \frac{1}{3}p_1. \end{aligned}$$

Profile C $n_1^p > 0$, $n_2^p > 0$, $n_1^{np} > 0$, $n_2^{np} = 0$. An analogous (to profile B) argument implies

$$\begin{aligned} n_1^p &= 1 + 2p_2 - 4p_1 \\ n_1^{np} &= 6p_1 - 1 - 2p_2 \\ n_2^p &= 1 - 2p_1 \end{aligned}$$

and this profile is part of an equilibrium if

$$\begin{aligned} 1 + 2p_2 - 4p_1 &\geq 0 \Longleftrightarrow p_1 \leq \frac{1}{4} + \frac{1}{2}p_2 \\ 1 - 2p_1 - 2p_2 &\geq 0 \Longleftrightarrow p_1 \leq \frac{1}{2} - p_2 \\ 6p_1 - 1 - 2p_2 &\geq 0 \Longleftrightarrow p_1 \geq \frac{1}{6} + \frac{1}{3}p_2. \end{aligned}$$

Profile D $n_1^p > 0$, $n_2^p > 0$, $n_1^{np} = 0$, $n_2^{np} = 0$. For this profile to be an equilibrium it must be that

1. $p_1 \leq n_1^p/2$
2. $p_2 \leq n_2^p/2$
3. $-p_1 - \frac{n_1^p}{2} = -p_2 - \frac{n_2^p}{2}$
4. $n_1^p + n_2^p = 1$

Conditions (3)+(4) imply that

$$\begin{aligned} n_1^p &= \frac{1}{2} + p_2 - p_1 \\ n_2^p &= \frac{1}{2} + p_1 - p_2, \end{aligned}$$

where $p_1 \leq n_1^p/2$ implies that $p_2 \geq 3p_1 - 1/2$ and $p_2 \leq n_2^p/2$ implies that $p_1 \geq 3p_2 - 1/2$.

Profile E $n_1^p > 0$, $n_2^p = 0$, $n_1^{np} > 0$, $n_2^{np} > 0$. For this profile to be part of an equilibrium customers must be indifferent between the priority service of provider 1, the regular service of provider 1 and the regular service of provider 2. That is

$$p_1 = \frac{n_1^p + n_1^{np}}{2} \ (E1)$$
$$-p_1 - \frac{n_1^p}{2} = -\frac{n_2^{np}}{2} \ (E2)$$
$$n_1^p + n_1^{np} + n_2^{np} = 1 \ (E3)$$
$$p_2 \geq \frac{n_2^{np}}{2} \ (E4).$$

$E1$ implies that $n_1^p + n_1^{np} = 2p_1$. Therefore, from $E3$ we get $n_2^{np} = 1 - 2p_1$. From $E2$ we get $\frac{n_1^p}{2} = \frac{n_2^{np}}{2} - p_1 = \frac{1}{2} - 2p_1 \iff n_1^p = 1 - 4p_1$. Therefore, $n_1^{np} = 2p_1 - n_1^p = 2p_1 - 1 + 4p_1 = 6p_1 - 1$. Putting the conditions together we get

$$\begin{aligned} n_1^p &= 1 - 4p_1 > 0 \iff p_1 < \frac{1}{4} \\ n_1^{np} &= 6p_1 - 1 > 0 \iff p_1 > \frac{1}{6} \\ n_2^{np} &= 1 - 2p_1 > 0 \\ p_2 &\geq \frac{1}{2} - p_1. \end{aligned}$$

Profile F $n_1^p = 0$, $n_2^p > 0$, $n_1^{np} > 0$, $n_2^{np} > 0$. An analogous argument (to profile E) implies that

$$\begin{aligned} n_2^p &= 1-4p_2>0 \Longleftrightarrow p_2<\frac{1}{4} \\ n_2^{np} &= 6p_2-1>0 \Longleftrightarrow p_2>\frac{1}{6} \\ n_1^{np} &= 1-2p_2>0 \\ p_1 &\geq \frac{1}{2}-p_2. \end{aligned}$$

Profile G $n_1^p=0$, $n_2^p=0$, $n_1^{np}>0$, $n_2^{np}>0$. Equilibrium conditions are

$$\begin{aligned} -\frac{n_1^{np}}{2} &= -\frac{n_2^{np}}{2} \\ n_1^{np}+n_2^{np} &= 1 \\ p_1 &\geq \frac{n_1^{np}}{2} \\ p_2 &\geq \frac{n_2^{np}}{2}. \end{aligned}$$

These conditions implies that $n_1^{np}=n_2^{np}=\frac{1}{2}$, $p_1\geq\frac{1}{4}$ and $p_2\geq\frac{1}{4}$.

Profile H $n_1^p>0$, $n_2^p=0$, $n_1^{np}=0$, $n_2^{np}>0$. For this profile to be an equilibrium it must be satisfy

$$\begin{aligned} -p_1-\frac{n_1^p}{2} &= -\frac{n_2^{np}}{2} \\ n_1^p+n_2^{np} &= 1 \\ p_1 &\leq \frac{n_1^p}{2} \\ p_2 &\geq \frac{n_2^{np}}{2}. \end{aligned}$$

The first two equalities imply that

$$\begin{aligned} n_1^p &= \frac{1}{2}-p_1>0 \\ n_2^{np} &= \frac{1}{2}+p_1>0. \end{aligned}$$

The conditions are $p_1\leq\frac{n_1^p}{2}\Longleftrightarrow p_1\leq\frac{1}{6}$ and $p_2\geq\frac{n_2^{np}}{2}\Longleftrightarrow p_2\geq\frac{1}{4}+\frac{1}{2}p_1$.

Profile I $n_1^p=0$, $n_2^p>0$, $n_1^{np}>0$, $n_2^{np}=0$. An analogous argument (to profile H) implies that in equilibrium

$$\begin{aligned} n_2^p &= \frac{1}{2}-p_2>0 \\ n_1^{np} &= \frac{1}{2}+p_2>0, \end{aligned}$$

and the conditions for this profile are $p_2\leq\frac{1}{6}$ and $p_1\geq\frac{1}{4}+\frac{1}{2}p_2$.

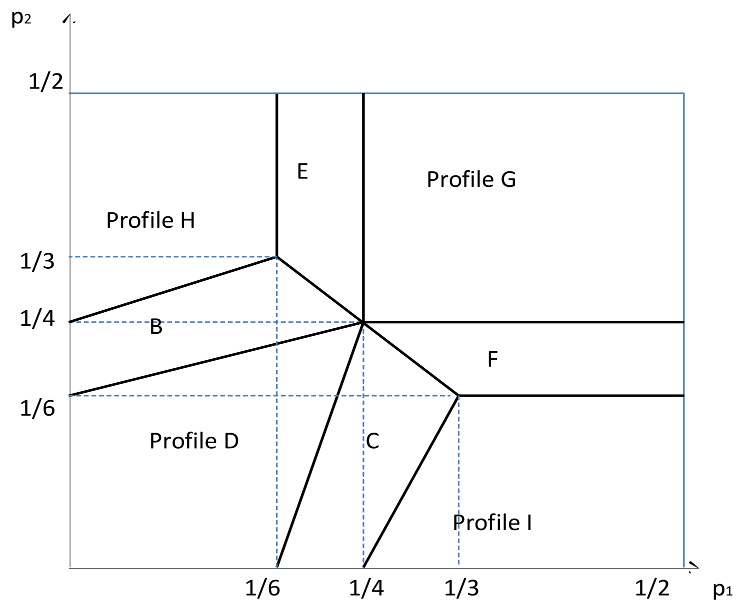

Figure 1: Profiles of agents' choices.

We can now plot all these profiles (profile A is the plotted diagonal line).

Now after calculating the equilibrium at the second stage (following price announcement of the providers) we can calculate the best responses of the firms at the first stage. We have to consider a few cases.

Case 1. $p_2 < 1/6$. In this case if $p_1 \leq \frac{1}{3}p_2 + \frac{1}{6}$, then we are in profile D and $n_1^p = \frac{1}{2} + p_2 - p_1$. Provider 1's maximization problem is

$$\begin{aligned} \max_{p_1} R &= p_1 \left(\frac{1}{2} + p_2 - p_1 \right) \\ s.t. p_1 &\leq \frac{1}{3}p_2 + \frac{1}{6}. \end{aligned}$$

Observe that R is concave in p_1 and reaches its maximum at $\frac{1}{2}p_2 + \frac{1}{4} > \frac{1}{3}p_2 + \frac{1}{6}$. Therefore, the optimal $p_1 = \frac{1}{3}p_2 + \frac{1}{6}$. If $p_1 > \frac{1}{3}p_2 + \frac{1}{6}$ and $p_1 \leq \frac{1}{4} + \frac{1}{2}p_2$, then we are in profile C and $n_1^p = 1 + 2p_2 - 4p_1$. Provider 1's maximization problem is

$$\begin{aligned} \max R &= p_1 (1 + 2p_2 - 4p_1) \\ s.t. p_1 &> \frac{1}{3}p_2 + \frac{1}{6}. \end{aligned}$$

Again, R is concave in p_1 and reaches its maximum at $\frac{1}{4}p_2 + \frac{1}{8} < \frac{1}{3}p_2 + \frac{1}{6}$. Therefore, the optimal price is $p_1 = \frac{1}{3}p_2 + \frac{1}{6}$. If $p_1 > \frac{1}{4} + \frac{1}{2}p_2$ then we are in profile I where $n_1^p = 0$, and we can conclude that for any $p_2 < 1/6$, provider 1's best response is $p_1 = \frac{1}{3}p_2 + \frac{1}{6}$.

Case 2. $1/4 > p_2 \geq 1/6$. If $p_1 \leq 3p_2 - \frac{1}{2}$, then we are in profile B and $n_1^p = 1 - 2p_2$ and $\frac{\partial R}{\partial p_1} = 1 - 2p_2 > 0$, and the optimal price is $p_1 = 3p_2 - \frac{1}{2}$. If $p_1 > 3p_2 - \frac{1}{2}$ and $p_1 < \frac{1}{2} - p_2$ then we know that the optimal price is $p_1 = \frac{1}{3}p_2 + \frac{1}{6}$ (which is the same as the optimal price in profiles D and C, as was shown in case 1). For $p_1 > \frac{1}{2} - p_2$ we are in profile F where $n_1^p = 0$. Therefore, we need to compare the revenue from $p_1 = 3p_2 - \frac{1}{2}$ and $n_1^p = 1 - 2p_2$ (which is $\left(3p_2 - \frac{1}{2}\right)(1 - 2p_2)$) with the revenue from $p_1 = \frac{1}{3}p_2 + \frac{1}{6}$ and $n_1^p = \frac{1}{2} + p_2 - p_1 = \frac{1}{2} + p_2 - \frac{1}{3}p_2 - \frac{1}{6} = \frac{1}{3} + \frac{2}{3}p_2$ (which is $\left(\frac{1}{3}p_2 + \frac{1}{6}\right)\left(\frac{1}{3} + \frac{2}{3}p_2\right)$). Therefore, for all $p_2 \leq 1/4$ we have $\left(\frac{1}{3}p_2 + \frac{1}{6}\right)\left(\frac{1}{3} + \frac{2}{3}p_2\right) > \left(3p_2 - \frac{1}{2}\right)(1 - 2p_2)$. Finally, we need to compare the revenues from this equilibrium with the revenues from the best equilibrium in profile A, in which the revenues are bounded by $(6p_2 - 1)\left(\frac{1}{2} - p_2\right)$. Since $\left(\frac{1}{2} - p_2\right)(6p_2 - 1) < \left(\frac{1}{3}p_2 + \frac{1}{6}\right)\left(\frac{1}{3} + \frac{2}{3}p_2\right)$ the best response also here is $p_1 = \frac{1}{3}p_2 + \frac{1}{6}$.

Case 3. $1/3 > p_2 \geq 1/4$. If $p_1 \leq 2p_2 - \frac{1}{2}$, then we are in profile H where $n_1^p = \frac{1}{2} - p_1$. Provider 1's maximization problem is

$$\begin{aligned} \max R &= p_1\left(\frac{1}{2} - p_1\right) \\ s.t. p_1 &\leq 2p_2 - \frac{1}{2}. \end{aligned}$$

Again, R is concave in p_1 and reaches its maximum at $p_1 = \frac{1}{4}$. Since $\frac{1}{4} > 2p_2 - \frac{1}{2}$, the optimal price in this region is $p_1 = 2p_2 - \frac{1}{2}$, and the revenues are $\left(2p_2 - \frac{1}{2}\right)(1 - 2p_2)$. If $\frac{1}{2} - p_2 > p_1 > 2p_2 - \frac{1}{2}$, then we are in profile B and $n_1^p = 1 - 2p_2$ and $\frac{\partial R}{\partial p_1} = 1 - 2p_2 > 0$, and the optimal price is $p_1 = \frac{1}{2} - p_2$ and the revenues are $\left(\frac{1}{2} - p_2\right)(1 - 2p_2)$. If $\frac{1}{4} > p_1 > \frac{1}{2} - p_2$, then we are in profile E with $n_1^p = 1 - 4p_1$. In profile E provider 1's maximization problem is

$$\begin{aligned} \max R &= p_1(1 - 4p_1) \\ s.t. \frac{1}{4} &\geq p_1 \geq \frac{1}{2} - p_2. \end{aligned}$$

Again, R is concave in p_1 and reaches its maximum at $p_1 = \frac{1}{8}$. Since $\frac{1}{4} \geq p_1 \geq \frac{1}{2} - p_2$, the optimal price in this region is $p_1 = \frac{1}{2} - p_2$ and the revenues are $\left(\frac{1}{2} - p_2\right)(4p_2 - 1)$. For $p_1 > \frac{1}{4}$ we are in profile G where $n_1^p = 0$. Therefore, to find the best response for the case of $1/3 > p_2 \geq 1/4$ we need to compare $\left(2p_2 - \frac{1}{2}\right)(1 - 2p_2)$ with $\left(\frac{1}{2} - p_2\right)(1 - 2p_2)$ and $\left(\frac{1}{2} - p_2\right)(4p_2 - 1)$. Among these candidates, $\left(\frac{1}{2} - p_2\right)(1 - 2p_2)$ generates the highest revenues. Finally, we need to compare these revenues with the revenues from the best equilibrium in profile A, which are smaller than $\left(\frac{1}{2} - p_2\right)(1 - 2p_2)$, which is the revenue in profile B, and hence the best response for $1/3 > p_2 \geq 1/4$ is $p_1 = \frac{1}{2} - p_2$.

Case 4. $1/2 > p_2 \geq 1/3$. If $p_1 \leq \frac{1}{6}$ we are in profile H where $n_1^p = \frac{1}{2} - p_1$.

Provider 1's maximization problem is

$$\begin{aligned} \max R &= p_1 \left(\frac{1}{2} - p_1 \right) \\ s.t. p_1 &\leq \frac{1}{6}. \end{aligned}$$

Again, R is concave in p_1 and reaches its maximum at $p_1 = \frac{1}{4}$. Since $\frac{1}{4} > \frac{1}{6}$, the optimal price in this region is $p_1 = \frac{1}{6}$ and the revenue is $\frac{1}{18}$. If $\frac{1}{4} \geq p_1 > \frac{1}{6}$ we are in profile E where $n_1^p = 1 - 4p_1$. In profile E provider 1's maximization problem is

$$\begin{aligned} \max R &= p_1 (1 - 4p_1) \\ s.t. \frac{1}{4} &\geq p_1 > \frac{1}{6}. \end{aligned}$$

Again, R is concave in p_1 and reaches its maximum at price $p_1 = \frac{1}{8}$. Since $\frac{1}{8} < \frac{1}{6}$, the optimal price in this region is $p_1 = \frac{1}{6}$ and the revenue is $\frac{1}{18}$. For $p_1 > \frac{1}{4}$ we are in profile G where $n_1^p = 0$. Therefore, the best response for $1/3 > p_2 \geq 1/4$ is $p_1 = \frac{1}{6}$.

We can now summarize the best response of provider 1 as

$$p_1 = \begin{cases} \frac{1}{3}p_2 + \frac{1}{6} & \text{if} \quad p_2 < 1/4 \\ \frac{1}{2} - p_2 & \text{if} \quad 1/3 > p_2 \geq 1/4 \\ \frac{1}{6} & \text{if} \quad 1/2 > p_2 \geq 1/3 \end{cases}.$$

Similarly, the best response of provider 2 is

$$p_2 = \begin{cases} \frac{1}{3}p_1 + \frac{1}{6} & \text{if} \quad p_1 < 1/4 \\ \frac{1}{2} - p_1 & \text{if} \quad 1/3 > p_1 \geq 1/4 \\ \frac{1}{6} & \text{if} \quad 1/2 > p_1 \geq 1/3 \end{cases}.$$

We plot these best responses in Figure 2.

Therefore, in the unique equilibrium of this game both providers announce prices $p_1 = p_2 = \frac{1}{4}$. ■

Proof of Proposition 12. Observe that if $p_1 \geq p_2 > 0$ then the customers' welfare in the market with priorities is given by

$$-\int_{c_1^*}^{\bar{c}} \left(p_1 + c\frac{1 - F(c_1^*)}{2} \right) f(c)dc - \int_{c_2^*}^{c_1^*} \left(p_2 + c\frac{F(c_1^*) - F(c_2^*)}{2} \right) f(c)dc$$
$$- \int_0^{c_2^*} c \left(1 - F(c_1^*) + \frac{n_1^{np}}{2} \right) f(c)dc,$$

while if no priority service is offered it is $-\frac{\mathbb{E}(c)}{4}$. We show that for any $p_1 \geq p_2 > 0$ holds

$$-\frac{\mathbb{E}(c)}{4} \geq -\int_{c_1^*}^{\bar{c}} \left(p_1 + c\frac{1 - F(c_1^*)}{2} \right) f(c)dc \tag{5}$$
$$- \int_{c_2^*}^{c_1^*} \left(p_2 + c\frac{F(c_1^*) - F(c_2^*)}{2} \right) f(c)dc - \int_0^{c_2^*} c \left(1 - F(c_1^*) + \frac{n_1^{np}}{2} \right) f(c)dc$$

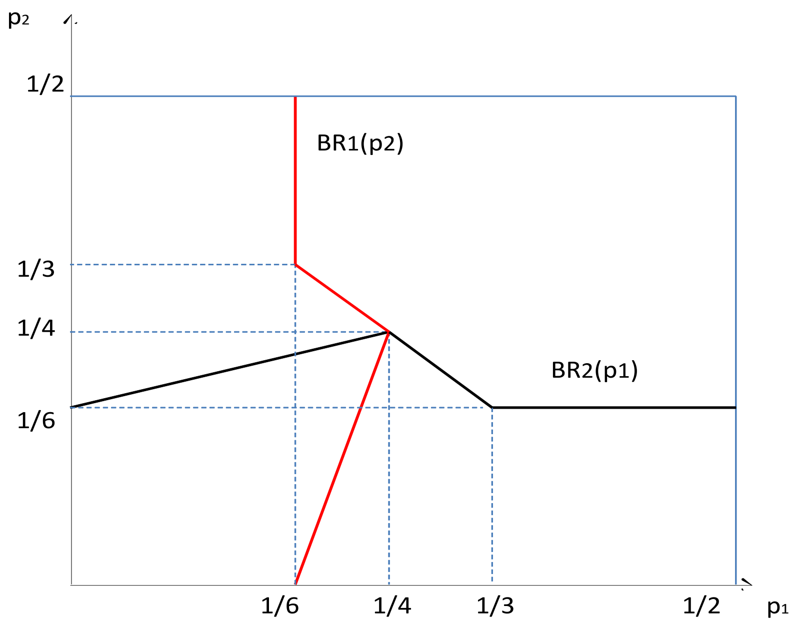

Figure 2: Best responses.

Observe that since $p_1 \geq p_2$ it implies that

$$F\left(c_1^*\right) - F\left(c_2^*\right) \geq 1 - F\left(c_1^*\right).$$

Furthermore, $F\left(c_1^*\right) > \frac{1}{2}$.

Plugging the expressions for p_1, p_2 and n_1^{np} we can rewrite the right hand side of the last inequality as

$$\begin{aligned}
&-\frac{1-F\left(c_1^*\right)}{2}\mathbb{E}(c) - \frac{F\left(c_1^*\right) - F\left(c_2^*\right) - 1 + F\left(c_1^*\right)}{2}\int_0^{c_1^*} cf(c)dc \\
&-\left(\frac{1-F\left(c_1^*\right)}{2} + \frac{F\left(c_2^*\right)}{4}\right)\int_0^{c_2^*} cf(c)dc - \frac{c_2^* - c_1^*}{2}\left(1 - F\left(c_1^*\right)\right) \\
&-\left(c_1^* - \frac{c_2^*}{2}\right)\left(F\left(c_1^*\right) - \frac{F\left(c_2^*\right)}{2}\right)\left(1 - F\left(c_1^*\right)\right) - c_2^*\left(F\left(c_1^*\right) - F\left(c_2^*\right)\right)\left(\frac{1-F\left(c_1^*\right)}{2} + \frac{F\left(c_2^*\right)}{4}\right)
\end{aligned}$$

The derivative of the last expression with respect to c_1^* is

$$\begin{aligned}
&f\left(c_1^*\right)\left(\frac{\mathbb{E}(c)}{2} - \int_0^{c_1^*} cf(c)dc + \frac{1}{2}\int_0^{c_2^*} cf(c)dc - c_1^*\left(1 - F\left(c_1^*\right)\right) + \frac{c_2^*}{2}\left(1 - F\left(c_2^*\right)\right)\right. \\
&\left.-\frac{1-F\left(c_1^*\right)}{f\left(c_1^*\right)}\left[F\left(c_1^*\right) - \frac{1}{2} - \frac{F\left(c_2^*\right)}{2}\right]\right)
\end{aligned}$$

We will first show that for the relevant parameters this derivative is negative. Plugging the expressions for $F(c) = c^\mu$ and $f(c) = \mu c^{\mu-1}$ gives

$$\begin{aligned}
& \frac{\mu}{\mu+1}\frac{1}{2} - \frac{\mu}{\mu+1}(c_1^*)^{\mu+1} - c_1^*\left(1-(c_1^*)^\mu\right) + \frac{1}{2}\frac{\mu}{\mu+1}(c_2^*)^{\mu+1} \\
& +\frac{1}{2}c_2^*\left(1-(c_2^*)^\mu\right) - \frac{1-(c_1^*)^\mu}{\mu(c_1^*)^{\mu-1}}\left((c_1^*)^\mu - \frac{1}{2} - \frac{(c_2^*)^\mu}{2}\right) \\
= & \left(1-\frac{1}{\mu+1}\right)\left(\frac{1}{2} - (c_1^*)^{\mu+1} + \frac{1}{2}(c_2^*)^{\mu+1}\right) - c_1^*\left(1-(c_1^*)^\mu\right) \\
& +\frac{1}{2}c_2^*\left(1-(c_2^*)^\mu\right) + \frac{1-(c_1^*)^\mu}{\mu(c_1^*)^{\mu-1}}\left(\frac{1}{2} - (c_1^*)^\mu + \frac{(c_2^*)^\mu}{2}\right) \\
= & \frac{1}{2} - c_1^* + \frac{1}{2}c_2^* - \frac{1}{\mu+1}\left(\frac{1}{2} - (c_1^*)^{\mu+1} + \frac{1}{2}(c_2^*)^{\mu+1}\right) \\
& +\frac{1-(c_1^*)^\mu}{\mu(c_1^*)^{\mu-1}}\left(\frac{1}{2} - (c_1^*)^\mu + \frac{(c_2^*)^\mu}{2}\right)
\end{aligned}$$

This expression increases in c_2^*. Hence, to show that the last expression is negative, it is enough to show it for c_2^* s.t.

$$(c_2^*)^\mu = 2(c_1^*)^\mu - 1.$$

Plugging the expression for c_2^*, we get that it is enough to show that

$$\frac{1}{2} - c_1^* + \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{1}{\mu}} - \frac{1}{\mu+1}\left(\frac{1}{2} - (c_1^*)^{\mu+1} + \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{\mu+1}{\mu}}\right) < 0 \text{ for any } (c_1^*)^\mu > \frac{1}{2} \text{ and } \mu \geq 1. \tag{6}$$

First observe that

$$\begin{aligned}
& \frac{1}{2} - c_1^* + \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{1}{\mu}} \\
= & -\left(c_1^* - \frac{1}{2}\right) + \left(\frac{1}{2}\right)^{\frac{\mu-1}{\mu}}\left((c_1^*)^\mu - \frac{1}{2}\right) < 0.
\end{aligned}$$

Where the last inequality holds since $\mu \geq 1$ and $1 > (c_1^*)^\mu \geq \frac{1}{2}$. If $\frac{1}{2} - (c_1^*)^{\mu+1} + \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{\mu+1}{\mu}} > 0$ for any $(c_1^*)^\mu > \frac{1}{2}$ and $\mu \geq 1$ we have inequality (6). Otherwise, since $\mu \geq 1$ it is enough to show that

$$\begin{aligned}
\frac{1}{2} - c_1^* + \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{1}{\mu}} - \frac{1}{2}\left(\frac{1}{2} - (c_1^*)^{\mu+1} + \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{\mu+1}{\mu}}\right) &< 0 \\
2\left(\frac{1}{2} - c_1^*\right) + 2(c_1^*)^\mu - 1 - \frac{1}{2} + (c_1^*)^{\mu+1} - \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{\mu+1}{\mu}} &< 0 \\
-2c_1^* + 2(c_1^*)^\mu - \frac{1}{2} + (c_1^*)^{\mu+1} - \frac{1}{2}\left(2(c_1^*)^\mu - 1\right)^{\frac{\mu+1}{\mu}} &< 0
\end{aligned}$$

where the last inequality holds since its left hand side is strictly increasing in c_1^* and for $c_1^* = 1$ its left hand side equals to 0.

Hence, to show (5) it is enough to show it for $(c_1^*)^\mu = \frac{1}{2}$ and $c_2^* = \left(2(c_1^*)^\mu - 1\right)^{\frac{1}{\mu}} = 0$. However, plugging these expressions into (5) gives that it holds as equality. ■

Duopoly with heterogeneous customers. Derivations of example 5.

We analyze here the duopoly equilibrium for heterogeneous customers. Given that $c_1^*(p_1, p_2)$ and $c_2^*(p_1, p_2)$ are cutoffs that satisfy conditions (1)-(3), the profit of provider 1 if it sets a price of p_1 for priority service and provider 2 sets price of p_2 is

$$\pi_1(p_1, p_2) = p_1\left(1 - F\left(c_1^*(p_1, p_2)\right)\right)$$

and, similarly, the profit of provider 2 is

$$\pi_2(p_1, p_2) = p_2\left(F\left(c_1^*(p_1, p_2)\right) - F\left(c_2^*(p_1, p_2)\right)\right).$$

For a profile (p_1, p_2) to be an equilibrium, it must be the case that[2]

$$\frac{\partial \pi_1(p_1, p_2)}{\partial p_1} = 0 \text{ and } \frac{\partial \pi_2(p_1, p_2)}{\partial p_2} = 0.$$

Hence the first-order conditions are

$$\begin{aligned}
\left(1 - F\left(c_1^*(p_1, p_2)\right)\right) - p_1 f\left(c_1^*(p_1, p_2)\right)\frac{\partial c_1^*(p_1, p_2)}{\partial p_1} &= 0 \\
\left(F\left(c_1^*(p_1, p_2)\right) - F\left(c_2^*(p_1, p_2)\right)\right) + p_2\left(f\left(c_1^*(p_1, p_2)\right)\frac{\partial c_1^*(p_1, p_2)}{\partial p_2} - f\left(c_2^*(p_1, p_2)\right)\frac{\partial c_2^*(p_1, p_2)}{\partial p_2}\right) &= 0.
\end{aligned}$$

We now calculate $\frac{\partial c_1^*(p_1,p_2)}{\partial p_1}$, $\frac{\partial c_1^*(p_1,p_2)}{\partial p_2}$ and $\frac{\partial c_2^*(p_1,p_2)}{\partial p_2}$ using the implicit function theorem. Denote by G_1 and G_2 as follows

$$\begin{aligned}
G_1(p_1, p_2, c_1, c_2) &= p_1 - p_2 - c_1\frac{2F(c_1) - F(c_2) - 1}{2} \\
G_2(p_1, p_2, c_1, c_2) &= p_2 - c_2\frac{2 - 2F(c_1) + F(c_2)}{4}.
\end{aligned}$$

The implicit function theorem implies that the derivatives $\frac{\partial c_j^*(p_1,p_2)}{\partial p_i}$ can be calculated from

$$\begin{pmatrix} \frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial c_2} \\ \frac{\partial G_2}{\partial c_1} & \frac{\partial G_2}{\partial c_2} \end{pmatrix} \times \begin{pmatrix} \frac{\partial c_1}{\partial p_i} \\ \frac{\partial c_2}{\partial p_i} \end{pmatrix} = -\begin{pmatrix} \frac{\partial G_1}{\partial p_i} \\ \frac{\partial G_2}{\partial p_i} \end{pmatrix},$$

conditional that

$$\det\begin{pmatrix} \frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial c_2} \\ \frac{\partial G_2}{\partial c_1} & \frac{\partial G_2}{\partial c_2} \end{pmatrix} \neq 0,$$

[2] In the case of a symmetric equilibrium the FOC are $\frac{\partial \pi_1(p,p)}{\partial p_1} \leq 0$ and $\frac{\partial \pi_2(p,p)}{\partial p_2} \geq 0$.

where the derivatives are evaluated at $(p_1, p_2, c_1^*(p_1,p_2), c_2^*(p_1,p_2))$. Applying Cramer's rule we get

$$\frac{\partial c_1}{\partial p_1} = -\frac{\det\begin{pmatrix} \frac{\partial G_1}{\partial p_1} & \frac{\partial G_1}{\partial c_2} \\ \frac{\partial G_2}{\partial p_1} & \frac{\partial G_2}{\partial c_2} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial c_2} \\ \frac{\partial G_2}{\partial c_1} & \frac{\partial G_2}{\partial c_2} \end{pmatrix}}, \quad \frac{\partial c_1}{\partial p_2} = -\frac{\det\begin{pmatrix} \frac{\partial G_1}{\partial p_2} & \frac{\partial G_1}{\partial c_2} \\ \frac{\partial G_2}{\partial p_2} & \frac{\partial G_2}{\partial c_2} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial c_2} \\ \frac{\partial G_2}{\partial c_1} & \frac{\partial G_2}{\partial c_2} \end{pmatrix}}, \quad \frac{\partial c_2}{\partial p_2} = -\frac{\det\begin{pmatrix} \frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial p_2} \\ \frac{\partial G_2}{\partial c_1} & \frac{\partial G_2}{\partial p_2} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial G_1}{\partial c_1} & \frac{\partial G_1}{\partial c_2} \\ \frac{\partial G_2}{\partial c_1} & \frac{\partial G_2}{\partial c_2} \end{pmatrix}}$$

where

$$\begin{aligned} \frac{\partial G_1}{\partial c_1} &= -\frac{2F(c_1) - F(c_2) - 1}{2} - c_1 f(c_1), \ \frac{\partial G_1}{\partial c_2} = c_1 \frac{f(c_2)}{2} \\ \frac{\partial G_2}{\partial c_1} &= c_2 \frac{f(c_1)}{2}, \ \frac{\partial G_2}{\partial c_2} = -\frac{2 - 2F(c_1) + F(c_2)}{4} - c_2 \frac{f(c_2)}{4} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial G_1}{\partial p_1} &= 1, \ \frac{\partial G_1}{\partial p_2} = -1 \\ \frac{\partial G_2}{\partial p_1} &= 0, \ \frac{\partial G_2}{\partial p_2} = 1. \end{aligned}$$

Assuming distribution function $F(c) = c^\mu$ gives the following first-order conditions

$$\begin{aligned} (1 - (c_1)^\mu) - p_1 \mu (c_1)^{\mu-1} \frac{\partial c_1^*(p_1,p_2)}{\partial p_1} &= 0 \\ ((c_1)^\mu - (c_2)^\mu) + p_2 \left(\mu (c_1)^{\mu-1} \frac{\partial c_1^*(p_1,p_2)}{\partial p_2} - \mu (c_2)^{\mu-1} \frac{\partial c_2^*(p_1,p_2)}{\partial p_2} \right) &= 0. \end{aligned}$$

with

$$\begin{aligned} p_1 &= c_2 \left[\frac{1 - (c_1)^\mu}{2} + \frac{(c_2)^\mu}{4} \right] + c_1 \frac{2(c_1)^\mu - (c_2)^\mu - 1}{2} \\ p_2 &= c_2 \left[\frac{1 - (c_1)^\mu}{2} + \frac{(c_2)^\mu}{4} \right] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial c_1}{\partial p_1} &= \frac{2 - 2(c_1)^\mu + (1+\mu)(c_2)^\mu}{\frac{2(1+\mu)(c_1)^\mu - (c_2)^\mu - 1}{2} (2 - 2(c_1)^\mu + (1+\mu)(c_2)^\mu) - \mu^2 (c_1 c_2)^\mu} \\ \frac{\partial c_1}{\partial p_2} &= \frac{2\mu c_1 (c_2)^{\mu-1} - (2 - 2(c_1)^\mu + (1+\mu)(c_2)^\mu)}{\frac{2(1+\mu)(c_1)^\mu - (c_2)^\mu - 1}{2} (2 - 2(c_1)^\mu + (1+\mu)(c_2)^\mu) - \mu^2 (c_1 c_2)^\mu} \\ \frac{\partial c_2}{\partial p_2} &= \frac{4(1+\mu)(c_1)^\mu - 2(c_2)^\mu - 2 - 2\mu c_2 (c_1)^{\mu-1}}{\frac{2(1+\mu)(c_1)^\mu - (c_2)^\mu - 1}{2} (2 - 2(c_1)^\mu + (1+\mu)(c_2)^\mu) - \mu^2 (c_1 c_2)^\mu} \end{aligned}$$

Example 5. Plugging $\mu = 1/2$ into the first order condition gives us

$$c_1 = 0.67336,\ c_2 = 0.34744.$$

This implies that the prices are

$$\begin{aligned} p_1 &= c_2\left(\frac{1-\sqrt{c_1}}{2}+\frac{\sqrt{c_2}}{4}\right)+c_1\frac{2\sqrt{c_1}-\sqrt{c_2}-1}{2} = 0.0.09978 \\ p_2 &= c_2\left(\frac{1-\sqrt{c_1}}{2}+\frac{\sqrt{c_2}}{4}\right) = 0.08237 \end{aligned}$$

and the providers' profits are

$$\begin{aligned} \pi_1 &= p_1\left(1-\sqrt{c_1}\right) = 0.0179 \\ \pi_2 &= p_2\left(\sqrt{c_1}-\sqrt{c_2}\right) = 0.019. \end{aligned}$$

The expected waiting time for regular service is

$$1 - F\left(c_1\right) + \frac{n_1^{np}}{2} = 0.35264.$$

Hence, the customers' surplus is

$$-0.35264\int_0^{c_2} cdc + \int_{c_2}^{c_1}\left(-p_2 - \frac{n_2^p}{2}c\right)dc + \int_{c_1}^{1}\left(-p_1 - \frac{n_1^p}{2}c\right)dc = -0.0878.$$

Without priority the customers' surplus is

$$-\frac{\mathbb{E}c}{4} = -\frac{\int_0^1 \frac{1}{2}\sqrt{s}ds}{4} = -0.083.$$

Proof of Proposition 15. If private service is offered, then the consumers' welfare is given by

$$\begin{aligned} &\int_{v^*}^{\bar{v}}\left(-p+vt_1\right)f\left(v\right)dv + \int_0^{v^*} v\hat{t}f(v)dv \\ = \;& v^*\left(\hat{t}-t_1\right)\left(1-F\left(v^*\right)\right) + t_1\int_{v^*}^{\bar{v}} vf\left(v\right)dv + \hat{t}\int_0^{v^*} vf(v)dv \\ = \;& v^*\left(\hat{t}-t_1\right)\left(1-F\left(v^*\right)\right) + \left(\hat{t}+t_1-\hat{t}\right)\int_{v^*}^{\bar{v}} vf\left(v\right)dv + \hat{t}\int_0^{v^*} vf(v)dv \\ = \;& v^*\left(\hat{t}-t_1\right)\left(1-F\left(v^*\right)\right) + \left(t_1-\hat{t}\right)\int_{v^*}^{\bar{v}} vf\left(v\right)dv + \hat{t}\mathbb{E}\left[v\right] \\ = \;& \left(t_1-\hat{t}\right)\int_{v^*}^{\bar{v}}\left(v-v^*\right)f\left(v\right)dv + \hat{t}\mathbb{E}\left[v\right]. \end{aligned}$$

We compare it to the consumers' welfare if private service is not available which is given by $\bar{t}\mathbb{E}[v]$. The regime without private system generates higher consumers' welfare if and only if

$$\begin{aligned}
\left(t_1 - \hat{t}\right)\int_{v^*}^{\bar{v}} (v - v^*) f(v)\, dv + \hat{t}\mathbb{E}[v] &\leq \bar{t}\mathbb{E}[v] \Longleftrightarrow \\
\left(t_1 - \hat{t}\right)\int_{v^*}^{\bar{v}} (v - v^*) f(v)\, dv &\leq \left(\bar{t} - \hat{t}\right)\mathbb{E}[v]
\end{aligned}$$

Observe that

$$\begin{aligned}
\bar{t} - \hat{t} &= \frac{t_1 + t_2 + \cdots + t_k}{k} - \frac{t_2 + \cdots + t_k}{k-1} = \frac{(k-1)t_1 - t_2 - \cdots - t_k}{k(k-1)} \\
t_1 - \hat{t} &= t_1 - \frac{t_2 + \cdots + t_k}{k-1} = \frac{(k-1)t_1 - t_2 - \cdots - t_k}{k-1}.
\end{aligned}$$

Hence we can rewrite the last inequality as follows

$$\begin{aligned}
\int_{v^*}^{\bar{v}} (v - v^*) f(v)\, dv &\leq \frac{1}{k}\mathbb{E}[v] \Longleftrightarrow \\
-(1 - F(v^*))\,\mathbb{E}[v] &\leq -\int_{v^*}^{\bar{v}} (v - v^*) f(v)\, dv.
\end{aligned}$$

For $v^* = 0$ and for $v^* = \bar{v}$ the last inequality holds as equality. Taking derivative of the left hand side wrt v^* we have $f(v^*)\mathbb{E}[v]$, the derivative of the right hand side is $\int_{v^*}^{\bar{v}} f(v)\, dv = 1 - F(v^*)$. The derivative of the right hand side is greater if and only if

$$f(v^*)\mathbb{E}[v] < 1 - F(v^*) \Longleftrightarrow \mathbb{E}[v] < \frac{1 - F(v)}{f(v)}.$$

The single-crossingness of $\mathbb{E}[v] - \frac{1-F(v)}{f(v)}$ implies that there exists v' s.t. for all $v < v'$ the derivative of the right hand side is greater than the derivative of the left hand side, while for all $v > v'$ the derivative of the left hand side is greater than the derivative of the right hand side. ■